Towards Analyzing Multimodality of Multiobjective Landscapes

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Introduction

Multiobjective optimization algorithms are particularly challenged by multimodality of the underlying landscape caused by interaction of objective functions. Thus, sophisticated Exploratory Landscape Analysis (ELA) features which are able to assess the level and type of multimodality have huge potential for understanding algorithm behaviour, automated algorithm selection and algorithm design.

Here, we lay the groundwork for constructing such experimental features systematically by providing formal definitions of multimodality in terms of distinguishing between local and global efficient sets.

Multimodality

Some topological notation in \( \mathbb{R}^m \) (decision space):

1. Let \( \mathcal{X} \subseteq \mathbb{R}^m \). The set \( \mathcal{X} \) is called connected if and only if there do not exist two open, disjoint subsets \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \) of \( \mathcal{X} \) such that \( \mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2 \) and \( \mathcal{X}_1 \cap \mathcal{X}_2 = \emptyset \).

2. Let \( \mathcal{X} \subseteq \mathbb{R}^m \). A subset \( \mathcal{C} \subseteq \mathcal{X} \) is a connected component of \( \mathcal{X} \) if \( \mathcal{C} \) is connected, and any subset of \( \mathcal{X} \) which is a strict subset of \( \mathcal{C} \) is not connected, and \( \mathcal{C} \) is non-empty.

Pareto concepts (very brief):

Let \( f : \mathcal{X} \rightarrow \mathbb{R}^m \) be a multiobjective function where \( \mathcal{X} \subseteq \mathbb{R}^m \) is the decision space. We will denote the component functions of \( f \) by \( f_i : \mathcal{X} \rightarrow \mathbb{R}, \quad i=1,\ldots,m \). A point \( x \in \mathcal{X} \) is called Pareto efficient or global efficient if there does not exist \( x' \in \mathcal{X} \) such that \( f_i(x') \leq f_i(x) \) for all \( i \). The subset of \( \mathcal{X} \) consisting of all the efficient points is denoted by \( \mathcal{X}_e \) and is called the efficient subset of \( \mathcal{X} \). The image of \( \mathcal{X}_e \) under \( f \) is called the Pareto front of \( f \).

Efficient points in \( \mathcal{X} \) and local efficient sets in the multiobjective case:

- A point \( x \in \mathcal{X} \) is called a locally efficient point of \( \mathcal{X} \) (or of \( f \)) if there is an open set \( U \subseteq \mathbb{R}^m \) with \( x \in U \) such that there is no point \( x' \in U \setminus \{x\} \) such that \( f(x') \leq f(x) \) for all \( i \). The subset of all the local efficient points of \( \mathcal{X} \) is denoted by \( \mathcal{X}_l \).

- A point \( x \in \mathcal{X} \) is called a global efficient point of \( \mathcal{X} \) (or of \( f \)) if there is no point \( x' \in \mathcal{X} \setminus \{x\} \) such that \( f(x') = f(x) \) for all \( i \). The subset of all the global efficient points of \( \mathcal{X} \) is called efficient set of \( \mathcal{X} \) and denoted by \( \mathcal{X}_e \).

- A subset \( \mathcal{C} \subseteq \mathcal{X} \) is a local efficient set of \( \mathcal{X} \) if \( \mathcal{C} \) is a connected component of \( \mathcal{X}_l \) and is the subset of \( \mathcal{X} \) which consists of the local efficient points of \( \mathcal{C} \).

Local and Global Pareto Fronts:

- A subset \( F \) of the image of \( f \) is a local Pareto front of \( f \) if there exists a local efficient set \( \mathcal{X}_l \) such that \( F = f(\mathcal{X}_l) \).

- The (global) Pareto Front (PF) of \( f \) is obtained by taking the image under \( f \) of the union of the connected components of the set of global efficient points of \( f \). If \( \mathcal{X}_e \) is connected, then the (global) Pareto front of \( f \) is also connected, provided \( f \) is continuous on \( \mathcal{X}_e \).

Analytics on Simple Mixed Sphere Problems

\begin{align*}
\mathcal{X}(N) & = \left\{ x \in \mathbb{R}^N \mid x \text{ is \textit{N}-dimensional} \right\} \\
g(x) & = \left( \frac{1}{2} \sum_{i=1}^{N} \left( x_i - x_i^0 \right)^2 \right)^{\frac{1}{2}}, \quad i = 1, \ldots, N
\end{align*}

functions \( g(x) \) define peaks with center \( x_i^0 \), depth \( N \), radius \( 1 \), and shape \( \alpha = \frac{1}{2} \). \( \alpha \) is the covariance matrix inverse. We focus on the simplified bi-objective case where each objective function consists of only one spherical peak. The efficient set is the line segment connecting \( x \) to \( f(x) \) for \( N = 2 \).

\begin{align*}
F_2 = \left\{ f(x) \mid x \in \mathcal{X}(2) \right\} \\
&= \left\{ \left( x_1, x_2 \right) \mid \frac{1}{2} \left( x_1 - x_1^0 \right)^2 + \frac{1}{2} \left( x_2 - x_2^0 \right)^2 \right\}
\end{align*}

The range of \( f(x) \) is \([-\sqrt{2}, \sqrt{2} \times \sqrt{2}, \sqrt{2}] \).

Experiments

- Evaluation of problem and algorithm characteristics
- MPM-2 Generator used to build simple and complex problems (examples presented here for \( m = 2 \))

<table>
<thead>
<tr>
<th>Count Characteristics</th>
<th>Scenario Simple</th>
<th>Scenario Complex</th>
</tr>
</thead>
<tbody>
<tr>
<td>optima (#) ( t_1, t_2 )</td>
<td>1 vs. 3</td>
<td>5 vs. 1</td>
</tr>
<tr>
<td>domination layers</td>
<td>3 vs. 19</td>
<td></td>
</tr>
<tr>
<td>connected components</td>
<td>3 vs. 30</td>
<td></td>
</tr>
<tr>
<td>sets connected to global efficient set fronts connected to global efficient front</td>
<td>2 vs. 12</td>
<td></td>
</tr>
<tr>
<td>local (global) efficient sets</td>
<td>4 vs. 11</td>
<td>107 vs. 11</td>
</tr>
</tbody>
</table>

- Apply two different local search strategies to the given problems:

<table>
<thead>
<tr>
<th>Algorithm Characteristics</th>
<th>Scenario Simple</th>
<th>Scenario Complex</th>
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<tbody>
<tr>
<td>Algorithms</td>
<td>HIGA-MO</td>
<td></td>
</tr>
<tr>
<td>Local Search Parameters</td>
<td></td>
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Problem Characteristics

- This work provides a thorough definition of multimodality for multiobjective optimization problems.
- Analytical and experimental approaches are presented which derive the global and local Pareto fronts.
- Mixed sphere test problems of different levels of multimodality are designed and the behavior of HIGA-MO and a stochastic local search variant are contrasted.
- Multimodality is a crucial factor determining the difficulty of a problem.
- Indicators are derived which allow to assess algorithm behaviour w.r.t. the detection of global and local Pareto fronts which can further be used for performance assessment and (later) algorithm selection.

Conclusion

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